Convex lattice polygons of fixed area with perimeter-dependent weights

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We study fully convex polygons with a given area, and variable perimeter length on square and hexagonal lattices. We attach a weight t^m to a convex polygon of perimeter m and show that the sum of weights of all polygons with a fixed area s varies as $s^{-\theta_{conv}}e^{K(t)\sqrt{s}}$ for large s and t less than a critical threshold t_c, where $K(t)$ is a t-dependent constant, and θ_{conv} is a critical exponent which does not change with t. Using heuristic arguments, we find that θ_{conv} is $1/4$ for the square lattice, but $-1/4$ for the hexagonal lattice. The reason for this unexpected nonuniversality of θ_{conv} is traced to existence of sharp corners in the asymptotic shape of these polygons.

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I. INTRODUCTION

The study of polygons is an important problem in lattice statistics [1]. It has been studied in the context of self-avoiding walks, and as a model of the shape transition in vesicles [2,3]. The problem is also related to the statistics of rare large finite clusters in the two dimensional percolation problem (see below). There has been considerable progress in counting exactly various subclasses of polygons weighted by area and perimeter (see [4,5] and references within). Recently, the exact critical scaling function of these polygons has also been found [6–11].

Convex polygons are an important subclass of polygons. They are defined as follows. The area enclosed by a polygon on a lattice is a simply connected set of elementary plaquettes or cells of the lattice. A polygon on is said to be column-convex in a given direction if all the plaquettes along any line in that direction are connected through plaquettes in the same line. The polygon is convex if it is column-convex in both the horizontal and vertical directions (see Fig. 1). A polygon on a hexagonal lattice is said to be convex if it is column-convex in all its three lattice directions (see Fig. 2).

Let $C_{m,s}$ be the number of convex polygons with perimeter m and area s. We define the generating function

$$C_s(t) = \sum_m C_{m,s} t^m. \tag{1}$$

For any finite s, this is a finite polynomial, and hence convergent. For large s, there exists a $t_c < 1$ such that for all $0 < t < t_c$, the leading contribution to the sum in Eq. (1) comes from polygons whose perimeter is of order $\sqrt{s}$. For the square lattice $t_c = 1/2$. For this sum, when t is sufficiently small, it is straightforward to prove upper and lower bounds that vary as an exponential of $\sqrt{s}$. It is expected that the leading correction to the exponential behavior is a power law,

$$C_s(t) \sim s^{-\theta_{conv}} e^{K(t)\sqrt{s}}, \quad s \to \infty, t < t_c, \tag{2}$$

where $K(t)$ is a t-dependent function, and θ_{conv} is a critical exponent. When t tends to zero, $K(t)$ tends to $C \ln(t)$. For the square lattice $C=4$, since the shape that minimizes the surface area is a square with perimeter $4\sqrt{s}$. The power-law exponent θ corresponding to other subclasses of polygons will be denoted by a suitable subscript.

In this paper, we calculate θ_{conv} for convex polygons on the square and hexagonal lattices by summing over all polygons with a fixed area and weighted by perimeter, and argue that θ_{conv} for the square lattice is $1/4$, but for the hexagonal lattice it is $-1/4$. We explain this difference by showing that the asymptotic shape of large convex polygons on square and hexagonal lattices consist of 4 and 6 cusps respectively. For

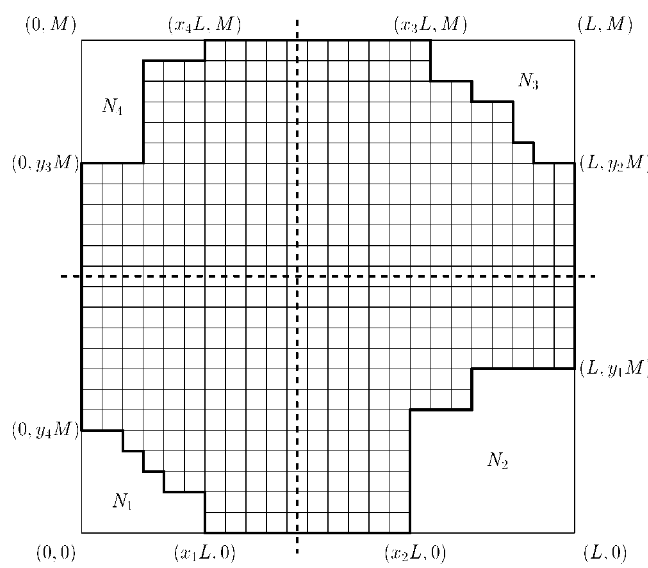

FIG. 1. A typical convex polygon on a square lattice and its bounding box is shown. All vertical and horizontal straight lines (dotted in the figure) intersect the polygon either 0 or 2 times. The convex polygon can be thought of as a rectangle from whose corners some squares have been removed by staircaselike paths.

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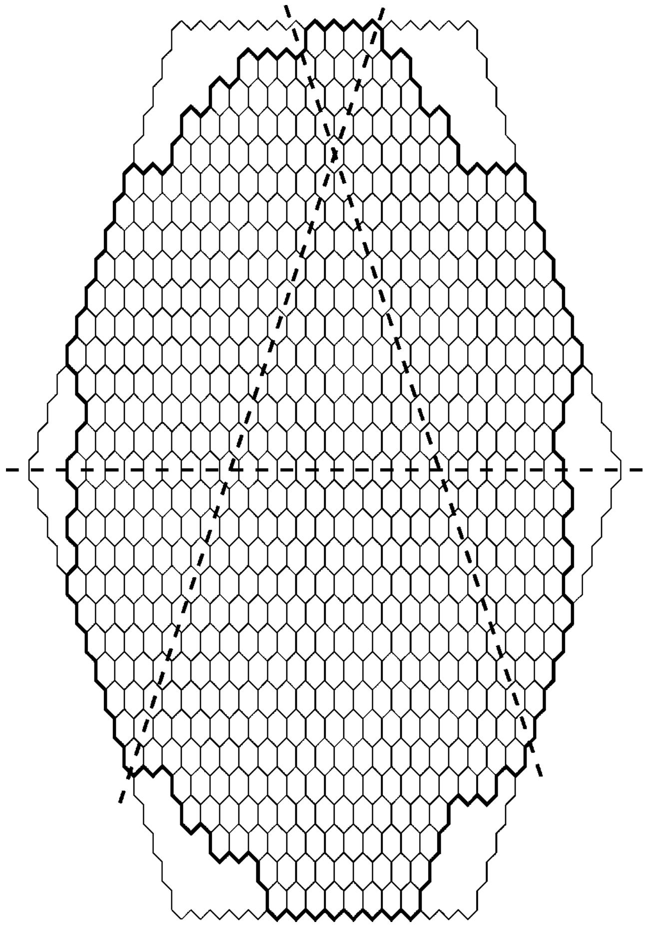

FIG. 2. A typical convex polygon on a hexagonal lattice is shown. Any straight line in the three lattice directions (shown as dotted lines) intersect the polygon at most twice. The convex polygon can be thought of as 6 blocks carved out by directed staircase-like paths from a bounding hexagon.

a polygon whose macroscopic shape has n cusps, we conjecture that the value of θ is $(5-n)/4$.

In the percolation problem (see [12,13] for an introduction) above the percolation threshold, the probability $\mathrm{Prob_p}(s)$ of finite clusters of size s in d-dimensions is expected to vary as [14,15]

$$\mathrm{Prob_p}(s) \sim s^{-\theta_{\mathrm{perc}}} \exp[-B(p)s^{(d-1)/d}], \quad s \to \infty. \qquad (3)$$

Here the exponent θ_{perc} is expected to be universal, same for all p above the critical percolation threshold. For these rare large clusters in two dimensions, the linear size of a cluster of s sites varies as $\sqrt{s}$. It has a few holes, and the external boundary of the cluster has overhangs. These are normally expected to be irrelevant. On ignoring holes, we can model percolation clusters by hole-less clusters, and $\mathrm{Prob_p}(s)$ would have the same qualitative behavior. In particular, we expect that $\theta_{perc}=\theta_{poly}$, where θ_{poly} is the value of the exponent θ corresponding to the generating function for all lattice polygons with a fixed area and weighted by perimeter.

The macroscopic shape of rare large finite clusters for $p > p_c$ is convex. Local fluctuations of the surface at a nonzero angle to the x axis can be well approximated by the fluctuations of a staircase path. As most of the surface of the cluster has a nonzero finite slope, one may expect that dominant contribution to $\mathrm{Prob_p}(s)$ comes from convex polygons. This would suggest that $\theta_{perc}=\theta_{poly}=\theta_{conv}$. Our results show that the second equality is wrong. In fact, θ_{conv} turns out to be lattice dependent. For the percolation problem, the presumably exact value of the lattice-independent exponent θ_{perc} has been calculated in all dimensions using techniques of continuum field theory, within the droplet model which ignores the holes and overhangs in the clusters [15]. In two dimensions $\theta_{perc}=5/4$ (corresponds to $n=0$).

We also mention that in the percolation problem below the critical threshold, $\mathrm{Prob_p}(s) \sim s^{-\theta'} \exp[-A(p)s]$ when $s \to \infty$. In this case θ' is given by the animal exponent. In two dimensions, these are described by the behavior of the function $C_s^{poly}(t)$ for $t > t_c$ with $C_s^{poly}(t) \sim s^{-\theta'} \exp(K'(t)s)$. The exact value of θ' for undirected animals is 1, 3/2, and 11/6 for $d=2$, 3, and 4, respectively [16]. The exponent θ' for the directed animals take on the value 1/2 and 5/6 for $d=2$ and 3 [17].

We now briefly review known results for convex polygons. For convex polygons on a square lattice, the exact two-variable generating function $C(t,z)$, defined as

$$C(t,z) = \sum_s C_s(t) z^s, \qquad (4)$$

was calculated by Lin [18] and Bousquet-Mélou [19,20]. It was shown that

$$C(t,z) = G + 2\sum_{m=2}^{\infty} g_m \sum_{n=1}^{m-1} t^{-2n} \sum_{p=0}^{\infty} f_{n+p} + \sum_{m=3}^{\infty} g_m S_m, \qquad (5)$$

where

$$g_m(t,z) = t^{2m} \sum_{n=1}^{\infty} (t^2 z)^n \prod_{k=1}^{n} (1-z^k)^{-2} [u_{m-1,n} - (2+z^n) u_{m-2,n} + (1+2z^n) u_{m-3,n} - z^n u_{m-4,n}],$$

$$u_{k,n}(z) = \sum_{r=0}^{k} \prod_{m_1=1}^{n+r} (1-z^{m_1}) \prod_{m_2=1}^{n+k-r} (1-z^{m_2}) \prod_{m_3=1}^{r} (1-z^{m_3})^{-1} \times \prod_{m_4=1}^{k-r} (1-z^{m_4})^{-1}, \quad k \geqslant 0,$$

$$G(t,z) = \sum_{m=1}^{\infty} g_m(t,z),$$

$$S_m(t,z) = \sum_{n=1}^{m-2} g_n t^{-2n} (m-n-1), \qquad (6)$$

$$f_m(t,z) = h_m + \sum_{n=2}^{m} S_{n+1} \left[\frac{t^2 z}{h_1' - h_1} [h_m(h_n' - h_m)] + \delta_{m,n} t^{2n+2} z^n \right],$$

$$h_n(t,z) = t^{2n+2}z^n\left(1 + \sum_{m=1}^{\infty}\frac{(-t^2)^m z^{m(m+1+2n)/2}}{\prod_{r=1}^{m}(1-z^r)(1-t^2z^r)}\right)\times\left(1 + \sum_{m=1}^{\infty}\frac{(-t^2)^m z^{m(m+1)/2}}{\prod_{r=1}^{m}(1-z^r)(1-t^2z^r)}\right)^{-1},$$

$$h_n'(t,z) = t^2z^n\left(1 + \sum_{m=1}^{\infty}\frac{(-t^2)^m z^{m(m+1+2n)/2}}{\prod_{r=1}^{m}(1-z^r)(t^2-z^r)}\right)\times\left(1 + \sum_{m=1}^{\infty}\frac{(-t^2)^m z^{m(m+1)/2}}{\prod_{r=1}^{m}(1-z^r)(t^2-z^r)}\right)^{-1}.$$

It is not easy to extract the asymptotic behavior of $C_s(t)$ for large s and fixed small value of t from the complicated expressions Eqs. (5) and (6).

The asymptotic behavior of the coefficient of t^m in Eq. (4), when $z>1$, was determined in Ref. [21]. In this case, the dominant contribution comes from the largest s possible, which is $z^{m^2/16}$ [3]. To be more specific, it was proved [21] that for fixed $z>1$

$$\sum_s C_{m,s}z^s = A(z)z^{m^2/16}[1 + (\rho^m)], \quad m \to \infty, \tag{7}$$

for some $\rho<1$. The function $A(z)$ was shown to behave as

$$A(z) \sim \frac{1}{4}\left(\frac{\epsilon}{2\pi}\right)^{3/2} e^{2\pi^2/(3\epsilon)} \quad \text{as } \epsilon = \ln(z) \to 0^+. \tag{8}$$

We can determine $C_s(t)$ from $\Sigma_s C_{m,s}z^s$ by

$$C_s(t) = \sum_m t^m \frac{1}{2\pi i}\oint \frac{dz}{z^{s+1}}\sum_s C_{m,s}z^s. \tag{9}$$

The above results in Eqs. (7) and (8) are valid when $\epsilon m \gg 1$. To do the integral in Eq. (9), we are interested in the limit when $\epsilon \to 0^-$ with $m \sim \sqrt{s} \sim 1/\epsilon$. It is not clear how to extend the results Eqs. (7) and (8) in this regime. However, if we *assume* that the results remain valid qualitatively in this regime also, and the limits of m large and ϵ small can be taken in reverse order, we can estimate $C_{s,t}$ by the method of steepest descent, assuming that the contour integral is dominated by the saddle point on the real line. This gives $C_s(t) \sim s^{-5/4}e^{\sqrt{s}K(t)}$. However, these assumptions are hard to justify. In fact, as we shall show later in the paper, the above answer is not right. This implies that in the region of interest, the asymptotic behavior is indeed different and not given by Eqs. (7) and (8).

The rest of the paper is organized as follows. In Sec. II, the exponent θ_{conv} is calculated for the square and hexagonal lattice. In Sec. III, the macroscopic shape of convex polygons is determined. In Sec. IV, the results are extended to subclasses of convex polygons. In Sec. V, the macroscopic shape of column-convex polygons is determined. Finally, we end with a summary and conclusions in Sec. VI.

II. CALCULATION OF THE EXPONENT θ_{conv}

Consider convex polygons on a square lattice. A convex polygon of a given perimeter can be visualized as a bounding rectangle of the same perimeter from whose corners some area has been removed by staircaselike paths (see Fig. 1). These staircase paths have the constraint that they cannot intersect each other. All convex polygons may then be generated by considering all possible rectangles.

Let $R(z,A,B)$ be a generating function such that the coefficient of z^s enumerates the number of staircase paths from $(0,A)$ to $(B,0)$ enclosing an area s. We then obtain

$$\sum_s C_s(t)z^s = \sum_{x_i,y_i,L,M} t^{2(L+M)}z^{LM}R(z^{-1},x_1L,y_4M)R(z^{-1},(1-x_3)\times L,(1-y_2)M)R(z^{-1},(1-x_2)\times L,y_1M)R(z^{-1},x_4L,(1-y_3)M), \tag{10}$$

where L and M is the length of the sides of the bounding rectangle of the convex polygon, and x_iL's and y_jM's denote the end points of the staircaselike paths (see Fig. 1). In writing down Eq. (10) we have ignored the case when the staircases at two opposite corners may intersect. This will only make an exponentially small correction and will not modify the exponent θ_{conv}. From the theory of partitions [22], it is known that

$$R(z,A,B) = z^{A+B-1}\frac{(z)_{A+B-2}}{(z)_{A-1}(z)_{B-1}}, \tag{11}$$

where

$$(z)_A = \prod_{k=1}^{A}(1-z^k). \tag{12}$$

The asymptotic behavior of $R(z,A,B)$ was worked out for some limiting cases in Ref. [24]. However, these rigorous results do not carry over to the limits that are of interest in this paper. Instead, we proceed as follows. The asymptotic behavior of the coefficient of z^s in $R(z,A,B)$ for large s can be calculated by the method of steepest descent. To evaluate $(z)_A$, we take logarithms on both sides of Eq. (12) and convert the resultant sum into an integral by using the Euler-Maclaurin sum formula [23]. This gives

$$(z)_A \sim \frac{1}{\sqrt{\epsilon}}\exp\left(\frac{1}{\epsilon}\int_{e^{-\epsilon A}}^{1} dx\frac{\ln(1-x)}{x}\right), \quad \epsilon = -\ln(z) \to 0. \tag{13}$$

Let the coefficient of z^s in $R(z,A,B)$ be denoted by $R_s(A,B)$. Then,

$$R_s(A,B) = \frac{1}{2\pi i}\oint\frac{R(z,A,B)}{z^{s+1}}. \tag{14}$$

We will evaluate this integral by the method of steepest descent. We make the assumption that the contour integral is dominated by the saddle point close to $z=1$ on the real line. This assumption is hard to justify as there are many singularities of the integrand near the saddle point. However, a similar assumption gives the right answer for unrestricted partitions. We would be interested in the limit when A and B vary as $\sqrt{s}$. Define $a=A/\sqrt{s}$ and $b=B/\sqrt{s}$ with a and b remaining finite as $s\to\infty$. Then,

$$R_s(A,B) \sim \frac{1}{s^{3/4}} \int d\alpha\, e^{\sqrt{s}g(\alpha,a,b)}, \tag{15}$$

where we made the substitution $z=e^{-\alpha/\sqrt{s}}$, and the function $g(\alpha,a,b)$ is given by

$$g(\alpha,a,b) = \alpha + \frac{1}{\alpha}\left(\int_{e^{-\alpha(a+b)}}^{1} du \frac{\ln(1-u)}{u} - \int_{e^{-\alpha a}}^{1} du \frac{\ln(1-u)}{u} - \int_{e^{-\alpha b}}^{1} du \frac{\ln(1-u)}{u}\right). \tag{16}$$

The function $g(\alpha,a,b)$ has a minimum at some $\alpha=\alpha_0$ where α_0 is a function of a and b. On doing the integral in Eq. (15) by the saddle point method, the power law factor gets modified by a factor $s^{-1/4}$. Thus we obtain

$$R_s(A,B) \sim \frac{1}{s} \exp[\sqrt{s} f(A/\sqrt{s},B/\sqrt{s})], \tag{17}$$

where

$$f(a,b) = g(\alpha_0,a,b), \tag{18}$$

with $g(\alpha,a,b)$ as in Eq. (16) and α_0 being that value of α which minimizes $g(\alpha,a,b)$. The function $f(a,a)$ increases monotonically from 0 to $\pi\sqrt{2/3}$ when a increases from 1 to ∞. The value at infinity, $f(\infty,\infty)$, corresponds to the result for unrestricted partitions [25]. Clearly, the function $f(a,b)$ is a monotonically increasing function in both its variables.

From now on, we will consider the case when all the distances in Fig. 1 scale as $\sqrt{s}$, i.e., $L=l\sqrt{s}$ and $M=m\sqrt{s}$. Also, each of the N_i's varies linearly with s, i.e., $N_i=n_i s$. Equation (10) then reduces to

$$\begin{aligned} C_s(t) \sim \int \prod_{i=1}^{4} (dx_i dy_i dn_i) dl dm (\sqrt{s})^{10} s^4 \delta\Big(s\Big(1+\sum n_i - lm\Big)\Big) \\ \times t^{2\sqrt{s}(l+m)} \frac{1}{s^4} \exp\left\{\sqrt{s}\left[\sqrt{n_1} f\left(\frac{x_1 l}{\sqrt{n_1}},\frac{y_4 m}{\sqrt{n_1}}\right) + \sqrt{n_2} f\left(\frac{(1-x_2)l}{\sqrt{n_2}},\frac{y_1 m}{\sqrt{n_2}}\right)\right]\right\} \\ \times \exp\left\{\sqrt{s}\left[\sqrt{n_3} f\left(\frac{(1-x_3)l}{\sqrt{n_3}},\frac{(1-y_2)m}{\sqrt{n_3}}\right) + \sqrt{n_4} f\left(\frac{x_4 l}{\sqrt{n_4}},\frac{(1-y_3)m}{\sqrt{n_4}}\right)\right]\right\}, \end{aligned} \tag{19}$$

where the $(\sqrt{s})^{10}$ factor is due to the scaling of the distances, the s^4 factor is due to scaling of the N_i's and s^{-4} factor is due to the power law term in the asymptotic formula for partitions. Thus, there is an overall power law factor s^5.

In the limit of large s, the integrals can be performed by the saddle point method. We first note that the shape that has maximum contribution to the integral will have the symmetry of the square lattice, i.e., the bounding box will be a square of side $x_0\sqrt{s}$ and each of the N_i's will be equal to $\beta_0 s$. Consider the integration over the variables x_2,y_2,x_3,y_3 about this shape. Due to the monotonic behavior of the scaling function $f(x,y)$, the integrand is maximum with respect to these four variables at the end points of their limits of integration, namely x_1,y_1,x_3,y_3, respectively. On doing the saddle point integration, the contribution to θ_{conv} in the power law prefactor from each integration is $s^{-1/2}$. Thus we are left with a power law term s^3. With respect to the remaining coordinate variables x_1,y_1,x_3,y_3,l,m, the integrand takes on it maximum value at a point in the interior of the region of integration, and each such integration contributes a factor $s^{-1/4}$ to the power law prefactor. Thus, after integrating over all the coordinates, a power law factor of $s^{3/2}$ remains.

Now, only the integrals over the n_i's remain to be done. Out of the four integrals, one of them integrates away the delta function contributing a factor s^{-1}, while each of the others contributes a factor $s^{-1/4}$ to the power law prefactor. Collecting together these terms, we obtain

$$C_s(t) \sim \frac{1}{s^{1/4}} \exp\left\{4\sqrt{s}\left[\sqrt{\beta_0} f\left(\frac{x_0}{2\sqrt{\beta_0}},\frac{x_0}{2\sqrt{\beta_0}}\right) + x_0 \ln(t)\right]\right\}. \tag{20}$$

By adding all the areas, we have $x_0^2=1+4\beta_0$. We compute the term in the exponential in Eq. (20) in Sec. III [see Eq. (29)]. Equation (20) implies that for convex polygons on a square lattice

$$\theta_{conv}^{sq} = \frac{1}{4}. \tag{21}$$

The above calculation of θ_{conv} can be summarized as follows. Consider a convex polygon constructed from a bounding box by n staircase paths ($n=4$ for square lattice). The end points of the staircase paths can slide along the bounding box, and each path contributes three coordinates to be integrated over. Out of these $3n$ coordinates, two of them are fixed to prevent over counting of polygons which are identical modulo translations. Thus there are a total of $(3n-2)$ coordinates, each one of them varying as $\sqrt{s}$, to be integrated over. Each staircase path also encloses an area, varying as s, that has to be integrated over. Finally, there is a contribution s^{-1} from each such area, corresponding to the enumeration of staircase paths with fixed ends and fixed area. Thus the integrand has an overall power law factor $s^{(3n-2)/2}$ to start with. On doing the integrations, the first n coordinate integrals contribute a factor $s^{-1/2}$ each as the maximum occurs at the end points of the integration limits, while the remaining $(2n-2)$ coordinates contribute $s^{-1/4}$ each. Thus, after the integration over the coordinates, the power law factor is $s^{(n-1)/2}$. The integrations over the areas have the following contributions. One of them integrates over the delta function, contributing s^{-1}, while each of the other contribute a factor $s^{-1/4}$. Taking these corrections into account, we obtain that θ_{conv} for a n-sided convex polygon is

$$\theta_{conv}^{n\text{-}sided} = \frac{5-n}{4}. \tag{22}$$

We recover the square lattice result [Eq. (21)] when $n=4$ in Eq. (22)

Consider now convex polygons on a hexagonal lattice (see Fig. 2). It is quite straightforward to carry out a similar analysis as was done for the square lattice. Equivalently, putting $n=6$ in the expression for θ_{conv} for n-sided convex polygons, we obtain

$$\theta_{conv}^{hex} = -\frac{1}{4}. \tag{23}$$

Equations (21) and (23) imply that θ_{conv} is not universal for convex polygons and takes on different values on different lattices.

III. MACROSCOPIC SHAPE OF CONVEX POLYGONS

The fact that θ_{conv} for the square and hexagonal lattices comes out different is somewhat unexpected. To understand the reason for this difference, and why this differs from the value $\theta_{perc}=5/4$ for percolation clusters, we need to look at the macroscopic shape of convex polygons. This can be done exactly using the Wulff construction [26]. Consider the case on the square lattice. The equilibrium curve is the one that extremizes the free energy functional

$$\mathcal{L}[y(x)] = \int_0^X dx\sigma(y')\sqrt{1+y'^2} - \frac{2\lambda}{\sqrt{s}}\int_0^X y dx, \tag{24}$$

where $y'=dy/dx$, $\sigma(y')$ is the orientation dependent surface tension and λ is a Lagrange multiplier. The equilibrium curve $y_0(x)$ satisfies the Euler-Lagrange equation

$$-\frac{d}{dx}\left(\frac{d}{dy'}[\sigma(y')\sqrt{1+y'^2}]\right) - \frac{2\lambda}{\sqrt{s}} = 0. \tag{25}$$

The equilibrium macroscopic shape is then obtained by minimizing $\mathcal{L}[y_0(x)]$ with respect to the end point X.

For convex polygons, it is easy to determine the slope dependent surface tension exactly. It has two contributions: one coming from the energy of the interface, and one from the entropy. For an interface having X horizontal and Y vertical steps, the energy per unit length is proportional to $|X|+|Y|$, and the number of configurations is $(|X|+|Y|)!/|X|!|Y|!$. This gives

$$\sigma(y')\sqrt{1+y'^2} = -(1+|y'|)\ln(1+|y'|) + |y'|\ln(|y'|) - (1+|y'|)\ln(t). \tag{26}$$

Following the above procedure, we obtain that the macroscopic shape of the staircase satisfies the equation

$$e^{-2\lambda|y|/\sqrt{s}} + e^{-2\lambda|x|/\sqrt{s}} = t^{-1}, \tag{27}$$

where the Lagrange multiplier λ is the negative root of

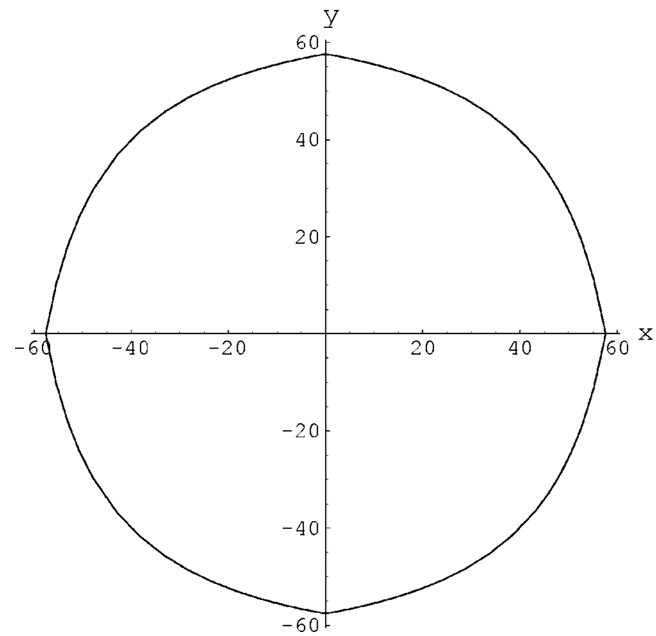

FIG. 3. The equilibrium shape of a convex polygon on a square lattice enclosing an area 10000 when the perimeter weight $t=0.15$ is shown.

$$\lambda^2 = \ln(t)\ln\left(\frac{t}{1-t}\right) + \int_t^{1-t} du \frac{\ln(1-u)}{u}. \tag{28}$$

At t tends to zero, the shape tends to the square $\max(|x|,|y|)=\sqrt{s}/2$. When t tends to $1/2$, then λ tends to zero and the shape tends to $|x|+|y|=\sqrt{s/2}$. When $t=1$, The shape Eq. (27) reduces to that for unrestricted partitions [27,28].

The term in the exponential of Eq. (20) can be calculated by substituting Eqs. (26)–(28) into Eq. (25). Doing so, we obtain

$$C_s(t) \sim \frac{1}{s^{1/4}}e^{\lambda\sqrt{s}}, \tag{29}$$

where λ is a function of t determined by Eq. (28). The equilibrium shape of a convex polygon enclosing an area 10000 when $t=0.15$ is shown in Fig. 3. The four staircase paths intersect each other at a finite angle. The reason why we see cusps in the macroscopic shape is the term proportional to $|y'|\ln(|y'|)$ in the expression for the direction dependent surface tension $\sigma(y')$. This singular term makes $\sigma(y')$ a local maximum at $y'=0$, which leads to a cusp. The macroscopic shape has four cusps due to the fourfold symmetry of the square lattice.

A similar analysis can be done for convex polygons on a hexagonal lattice. The surface energy $\sigma(y')$ has qualitatively the same behavior as for the square lattice. The sixfold symmetry of the hexagonal lattice results in 6 cusps for the hexagonal convex polygons.

For ordinary percolation, the continuum theory calculation [15] gives $\theta_{perc}=5/4$. On the other hand, θ_{conv} for a n-sided convex polygon takes on the value $5/4-n/4$. In addition, the macroscopic shape of a n-sided convex polygon has n cusps. These cusps are not expected to appear in the

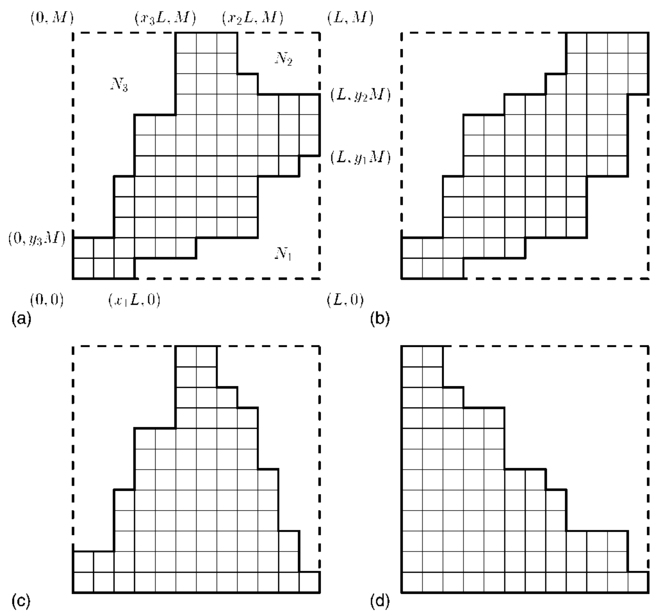

FIG. 4. Examples of (a) a directed convex polygon, (b) a staircase polygon, (c) a pyramidal polygon, and (d) Ferrers diagram on a square lattice.

macroscopic shape of percolation clusters. One would presume that on going beyond the convex polygons approximation, these cusps would disappear, each contributing a certain factor to the power law. Thus, putting $n=0$ in Eq. (22), we recover the result for percolation.

We can similarly determine the value of $\theta_{dir\ perc}$ for two-dimensional directed percolation (see [29] for definition and an introduction). Consider directed percolation above the percolation threshold. Let the infinite cluster have a finite opening angle $\pi/2-2\gamma$, where γ is a function of p. Then, the surface tension for surfaces which have slopes $\tan(\gamma)$ and $\tan(3\pi/2-\gamma)$ is zero. Due to these local minima, and hence a maximum at zero slope, the macroscopic shape of finite directed percolation has a cusp at the origin with an opening angle $\pi/2-2\gamma$. Thus, $\theta_{dir\ perc}$ for directed percolation is obtained by substituting $n=1$ in Eq. (22), yielding

$$\theta_{dir\ perc} = 1 \quad \text{in 2 dimensions}. \tag{30}$$

IV. SUBCLASSES OF CONVEX POLYGONS

In this section, we extend the results to subclasses of convex polygons. A directed convex polygon on a square lattice is a convex polygon for which the lower left corner of the bounding rectangle is also a vertex of the polygon [see Fig. 4(a)]. As for convex polygons, the area and perimeter weighted generating function for directed convex polygons is known [19,20]. We now determine the exponent θ in exactly the same way as was done for convex polygons.

Consider a directed convex polygon. The contribution to the power law prefactor from the various steps in the power counting is as follows. (1) The integrand initially has a power law factor $(\sqrt{s})^8$. (2) Integration over x_1, y_1, x_3, y_3 contributes $s^{-1/2}$ each to the power law factor. (3) Integration over x_2, y_2, l, m contributes $s^{-1/4}$ each to the power law factor. (4) Integration over N_1, N_2, N_3 contributes $(s^{-1/4})^2 s^{-1}$ to the power law factor. Collecting together the various terms, we obtain

$$\theta_{dir\ conv} = \frac{1}{2}, \tag{31}$$

where $\theta_{dir\ conv}$ is the θ corresponding to directed convex polygons. The macroscopic shape of the directed convex polygon has three cusps. Not surprisingly, substituting $n=3$ in Eq. (22) gives the result in Eq. (31).

The other subclasses of convex polygons that we study are staircase polygons, pyramidal polygons, and Ferrers diagrams. Staircase polygons are convex polygons for which both the lower left and upper right corners of the bounding rectangle are vertices of the polygon. Pyramidal polygons are convex polygons for which both the lower left and lower right corners of the bounding rectangle are vertices of the polygon. Ferrers diagrams are convex polygons for which the lower left, lower right, and upper left corners of the bounding rectangle are vertices of the polygons. Examples of the polygons are shown in Figs. 4(b)–4(d), respectively. The area and perimeter generating function of staircase polygons [30,31], pyramidal polygons [27] and Ferrers diagram [22] are known. The exponent θ can be calculated for each one of them as before. Proceeding on the same lines, we obtain

$$\theta_{stair} = \frac{3}{4}, \tag{32}$$

$$\theta_{pyramid} = \frac{1}{2}, \tag{33}$$

$$\theta_{Ferrer} = \frac{1}{2}. \tag{34}$$

These correspond to 2, 3 and 3 cusps, respectively, in the macroscopic equilibrium shape of these polygons.

V. COLUMN CONVEX POLYGONS

In this section, we determine the equilibrium shape of column-convex polygons and show that it has two cusps. An example of a column-convex polygon is shown in Fig. 5(a). The area and perimeter weighted generating function for column-convex polygons is known [31]. However, as for convex polygons, it is difficult to extract from it the asymptotic behavior of fixed area polygons.

We first calculate the angle dependent surface tension $\sigma_r(\gamma)$, where $y'=\tan(\gamma)$, for column-convex polygons. This analysis is similar to that done for directed polymers [32]. Consider all possible directed walks from (0,0) to (x,y). Then, the sum over all weighted paths is

$$e^{-x\,\sec(\gamma)\sigma_r(\gamma)} = \sum_{y_1,\ldots,y_x} \delta\left(\sum_{i=1}^{x} y_i - y\right) \prod_{i=1}^{x} t^{1+|y_i|}, \tag{35}$$

where δ is the usual Kronecker delta function. Taking Laplace transform with respect to y, we obtain independent

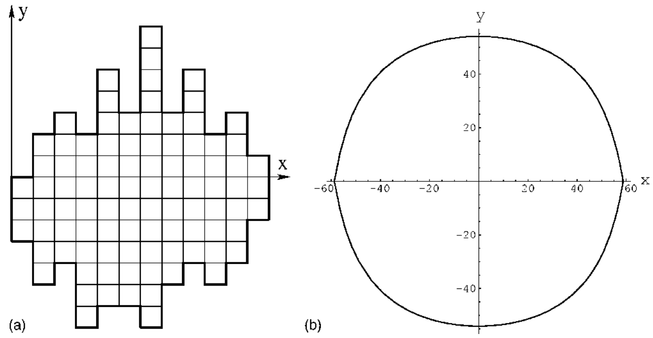

FIG. 5. (a) A column-convex polygon on a square lattice. Any line in the vertical direction intersects the polygon at either zero or two points. (b) The equilibrium macroscopic shape of a column-convex polygon on a square lattice enclosing an area 10000, when $t=0.15$.

summations over y_i. These are easily done giving

$$\sigma_r(y')\sqrt{1+y'^2} = y' \ln(z_0) + \ln \frac{(1-tz_0)(1-tz_0^{-1})}{t(1-t^2)}, \quad (36)$$

where

$$z_0 = \frac{(1+t^2)y' + \sqrt{(1-t^2)^2 y'^2 + 4t^2}}{2t(1+y')}. \quad (37)$$

We see that $\sigma_r(y')$ is now a smooth function of y' for y' near zero. For convex polygons, a surface with average orientation $y'=0$ cannot have any fluctuations, as the height fluctuations in the y direction become the disallowed overhangs in the x direction. This leads to the singularity near $y'=0$ in the expression for orientation dependent surface tension for convex polygons.

To construct the equilibrium shape of the polygon, we need to find the $y(x)$ satisfying the Euler Lagrange equation [see Eq. (25)] with σ_r and a Lagrange multiplier λ_r. The curve $y(x)$ satisfies the boundary condition $y(-X/2)=0$ and $y(X/2)=0$. Solving, we find that the shape of the polygon is given by

$$e^{2\lambda_r y/\sqrt{s}} = 4ct \sinh\left(\frac{\ln(t)}{2} - \frac{\lambda_r x}{\sqrt{s}}\right)\sinh\left(\frac{\ln(t)}{2} + \frac{\lambda_r x}{\sqrt{s}}\right), \quad (38)$$

where c is a constant, $X=g(c)\sqrt{s}/\lambda_r$, and

$$g(c) = \ln\left[\frac{c(1+t^2) - 1 + \sqrt{[1-c(1-t^2)]^2 - 4c^2t^2}}{2ct}\right]. \quad (39)$$

The Lagrange multiplier λ_r is fixed by the constraint that $\int_{-X/2}^{X/2} y dx = s/2$. We obtain λ_r as a function of c to be

$$\lambda_r^2 = \int_0^{g(c)} dz \ln[c(1-te^{-z})(1-te^{z})]. \quad (40)$$

The value of c is chosen to be the one that minimizes the total surface free energy. For the curve Eq. (38), the total surface energy $F(c,t)$ is

$$F(c,t) = 2\lambda_r\sqrt{s} - \frac{\sqrt{s}}{\lambda_r} g(c)\ln[ct(1-t^2)]. \quad (41)$$

Minimizing Eq. (41) with respect to c, we obtain

$$c = \frac{1}{t(1-t^2)}. \quad (42)$$

Equations (38), (40), and (42) describe the equilibrium shape of column-convex polygons. In Fig. 5(b), the shape when $t=0.15$ is shown. It has two cusps. Thus we would conclude from Eq. (22) that

$$\theta_{col\ conv} = \frac{3}{4}. \quad (43)$$

The height fluctuations of the column-convex polygons become overhangs when viewed after rotation by $\pi/2$. On introducing such overhangs, two of the four cusps that were present in the shape of convex polygons vanished. Thus, one would expect that if overhangs in the horizontal direction were also allowed as in self avoiding polygons, then there would be no cusps, and

$$\theta_{poly} = \frac{5}{4}. \quad (44)$$

Finally, we note that the macroscopic shape of column-convex polygons becomes unstable when $\sigma_r(0)=0$. The smallest absolute value of t at which this occurs is

$$t_c = \sqrt{2} - 1. \quad (45)$$

This value matches with the previously obtained value for t_c [33,34].

VI. SUMMARY AND CONCLUSION

To summarize, we studied fixed area convex polygons weighted by their perimeter on square and hexagonal lattices. Based on heuristic arguments, the exponent θ_{conv} as defined in Eq. (2) was found to be $1/4$ for the square lattice and $-1/4$ for the hexagonal lattice. This discrepancy was traced to the presence of cusps in the macroscopic shape of convex polygons. We argued that for a polygon whose macroscopic shape has n cusps has $\theta_n=(5-n)/4$. While our arguments are nonrigorous, we conjecture that these results are correct. For polygons, one expects that the macroscopic shape has no cusps. Indeed, putting $n=0$ in Eq. (22), we recover the result $\theta_{perc}=5/4$ obtained for percolation clusters [15]. We also extended these results to directed percolation in two dimensions ($n=1$), directed convex polygons ($n=3$), staircase polygons ($n=2$), pyramidal polygons ($n=3$), Ferrers diagram, ($n=3$) and column convex polygons ($n=2$).

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